JEE ADVANCED

Single Correct Answer Type

1. If $f(x) = \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x\to\infty} f(x)$ is

a. 0 b. ∞
c. 1 d. none of these
(IIT-JEE 1979)

2. If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x\to 1} \dfrac{G(x) - G(1)}{x-1}$ is

a. $\dfrac{1}{24}$ b. $\dfrac{1}{5}$
c. $-\sqrt{24}$ d. none of these
(IIT-JEE 1983)

3. $\lim_{n\to\infty} \left\{ \dfrac{1}{1-n^2} + \dfrac{2}{1-n^2} + \cdots + \dfrac{n}{1-n^2} \right\}$ is equal to

a. 0 b. $-\dfrac{1}{2}$
c. $\dfrac{1}{2}$ d. none of these
(IIT-JEE 1984)

4. If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ is

a. 1 b. 0
c. −1 d. none of these
(IIT-JEE 1985)

5. The value of $\lim_{x\to 0} \dfrac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x}$ is

a. 1 b. −1
c. 0 d. none of these
(IIT-JEE 1991)

6. $\lim_{x\to 1} \dfrac{\sqrt{1 - \cos 2(x-1)}}{x-1}$

a. exists and it equals $\sqrt{2}$
b. exists and it equals $-\sqrt{2}$
c. does not exist because $x - 1 \to 0$
d. does not exist because the left-hand limit is not equal to the right-hand limit **(IIT-JEE 1998)**

7. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$, for every value of θ, then

a. $b_0 = 1, b_1 = 3$ b. $b_0 = 0, b_1 = n$
c. $b_0 = -1, b_1 = n$ d. $b_0 = 0, b_1 = n^2 - 3n + 3$
(IIT-JEE 1998)

8. $\lim_{x\to 0} \dfrac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is equal to

a. 2 b. −2
c. 1/2 d. −1/2 **(IIT-JEE 1999)**

9. For $x \in R$, $\lim_{x\to\infty} \left(\dfrac{x-3}{x+2}\right)^x$ is equal to

a. e b. e^{-1}
c. e^{-5} d. e^5 **(IIT-JEE 2000)**

10. $\lim_{x\to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

a. $-\pi$ b. π
c. $\pi/2$ d. 1 **(IIT-JEE 2001)**

11. The integer n for which $\lim_{x\to 0} \dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number is

a. 1 b. 2
c. 3 d. 4 **(IIT-JEE 2002)**

12. If $\lim_{x\to 0} \dfrac{\{(a-n)nx - \tan x\} \sin nx}{x^2} = 0$, where n is nonzero real number, then a is

a. 0 b. $\dfrac{n+1}{n}$
c. n d. $n + \dfrac{1}{n}$ **(IIT-JEE 2003)**

13. The value of $\lim_{x\to 0}((\sin x)^{1/x} + (1+x)^{\sin x}) = 0$, where $x > 0$, is

a. 0 b. −1
c. 1 d. 2 **(IIT-JEE 2006)**

14. Let $g(x) = \dfrac{(x-1)^n}{\log \cos^m (x-1)}; 0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim\limits_{x\to 1^+} g(x) = p$, then

a. $n = 1, m = 1$
b. $n = 1, m = -1$
c. $n = 2, m = 2$
d. $n > 2, m = n$

(IIT-JEE 2008)

15. If $\lim\limits_{x\to 0}[1 + x\ln(1+b^2)]^{1/x} = 2b\sin^2\theta, b > 0,$

smf $\theta \in (-\pi, \pi]$, then the value of θ is

a. $\pm\dfrac{\pi}{4}$
b. $\pm\dfrac{\pi}{3}$
c. $\pm\dfrac{\pi}{6}$
d. $\pm\dfrac{\pi}{2}$ **(IIT-JEE 2011)**

16. If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+x+1}{x+1} - ax - b\right) = 4$, then

a. $a = 1, b = 4$
b. $a = 1, b = -4$
c. $a = 2, b = -3$
d. $a = 2, b = 3$

(IIT-JEE 2012)

17. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$ where $a > -1$. Then $\lim\limits_{a\to 0^+} \alpha(a)$ and $\lim\limits_{a\to 0^+} \beta(a)$ are

a. $-\dfrac{5}{2}$ and 1
b. $-\dfrac{1}{2}$ and -1
c. $-\dfrac{7}{2}$ and 2
d. none of these

(IIT-JEE 2012)

Multiple Correct Answers Type

1. Let $L = \lim\limits_{x\to 0}\dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$, $a > 0$. If L is finite, then

a. $a = 2$
b. $a = 1$
c. $L = \dfrac{1}{64}$
d. $L = \dfrac{1}{32}$ **(IIT-JEE 2009)**

Integer Answer Type

1. The largest value of the non-negative integer a for which

$\lim\limits_{x\to 1}\left\{\dfrac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$ is

(JEE Advanced 2014)

2. Let m and n be two positive integers greater than 1. If

$\lim\limits_{\alpha\to 0}\dfrac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\dfrac{e}{2}$, then the value of $\dfrac{m}{n}$ is

(JEE Advanced 2015)

Fill in the Blanks Type

1. $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2} =$ ______ . **(IIT-JEE 1984)**

2. If $f(x) = \begin{cases}\sin x, & x \neq n\pi, n \in I \\ 2, & \text{otherwise}\end{cases}$

and $g(x) = \begin{cases} x^2 + 1, & x \neq 0, \\ 4, & x = 0 \\ 5, & x = 2\end{cases}$

then $\lim\limits_{x\to 0} g\{f(x)\}$ is =

______ . **(IIT-JEE 1986)**

3. $\lim\limits_{x\to -\infty}\left[\dfrac{x^4\sin\left(\dfrac{1}{x}\right) + x^2}{(1 + |x|^3)}\right] =$ ______ . **(IIT-JEE 1987)**

4. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC, then triangle ABC has perimeter $P = 2\left(\sqrt{2hr - h^2} + \sqrt{2hr}\right)$ and area $A =$ ______ and also

$\lim\limits_{h\to 0}\dfrac{A}{P^3} =$ ______ . **(IIT-JEE 1989)**

5. $\lim\limits_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4} =$ ______ . **(IIT-JEE 1990)**

6. $\lim\limits_{x\to 0}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{1/x^2} =$ ______ . **(IIT-JEE 1996)**

7. $\lim\limits_{h\to 0}\dfrac{\ln(1+2h) - 2\ln(1+h)}{h^2} =$ ______ . **(IIT-JEE 1997)**

True/False Type

1. If $\lim\limits_{x\to a}[f(x)g(x)]$ exists, then both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. **(IIT-JEE 1981)**

Subjective Type

1. Evaluate $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}, (a \neq 0)$.

(IIT-JEE 1978)

2. $f(x)$ is the integral of $\dfrac{2\sin x - \sin 2x}{x^3}, x \neq 0$. Find

$\lim\limits_{x\to 0} f'(x)\left[\text{where } f'(x) = \dfrac{df(x)}{dx}\right]$. **(IIT-JEE 1979)**

3. Evaluate $\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$.

(IIT-JEE 1980)

4. Use the formula $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$ to find $\lim_{x \to 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$.

(IIT-JEE 1982)

5. Find $\lim_{x \to 0} \{\tan(\pi/4 + x)\}^{1/x}$. **(IIT-JEE 1993)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. c.	**2.** d.	**3.** b.	**4.** d.
5. d.	**6.** d.	**7.** b.	**8.** c.
9. c.	**10.** b.	**11.** c.	**12.** d.
13. c.	**14.** c.	**15.** d.	**16.** b.
17. b.			

Multiple Correct Answers Type

1. a., c.

Integer Answer Type

1. (2) **2.** (2)

Fill in the Blanks Type

1. $\frac{2}{\pi}$ **2.** 1 **3.** -1

4. $A = h\sqrt{2rh - h^2}, \frac{1}{128r}$ **5.** e^5

6. e^2 **7.** -1

True/False Type

1. False

Subjective Type

1. $\frac{2}{3\sqrt{3}}$ **2.** 1 **3.** $a^2 \sin a + 2a \sin a$

4. $2 \ln 2$ **5.** e^2

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. c. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\sqrt{\dfrac{1-\dfrac{\sin x}{x}}{1+\dfrac{\cos^2 x}{x}}} = \sqrt{\dfrac{1-0}{1+0}} = 1$

2. d. $\lim_{x\to 1}\dfrac{G(x)-G(1)}{x-1}\lim_{x\to 1}\dfrac{-\sqrt{25-x^2}-(-\sqrt{24})}{x-1}$

$$= \lim_{x\to 1}\frac{\sqrt{24}-\sqrt{25-x^2}}{x-1}\times\frac{\sqrt{24}+\sqrt{25-x^2}}{\sqrt{24}+\sqrt{25-x^2}}$$

$$= \lim_{x\to 1}\frac{x^2-1}{(x-1)\left[\sqrt{24}+\sqrt{25-x^2}\right]}$$

$$= \lim_{x\to 1}\frac{x+1}{\sqrt{24}+\sqrt{25-x^2}}$$

$$= \frac{2}{2\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

Alternative Method:

$$\lim_{x\to 1}\frac{G(x)-G(1)}{x-1}$$

$$= \lim_{x\to 1}\frac{-\sqrt{25-x^2}-(-\sqrt{24})}{x-1} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{\dfrac{x}{\sqrt{25-x^2}}}{1} \quad \text{(Applying L'Hospital's Rule)}$$

$$= \frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}}$$

3. b. $\lim_{n\to\infty}\left(\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\cdots+\dfrac{n}{1-n^2}\right)$

$$= \lim_{n\to\infty}\frac{1+2+3+\cdots+n}{1-n^2}$$

$$= \lim_{n\to\infty}\frac{\dfrac{n(n+1)}{2}}{1-n^2}$$

$$= \lim_{n\to\infty}\frac{1+\dfrac{1}{n}}{2\left[\dfrac{1}{n^2}-1\right]} = -1/2$$

4. d. The given function is

$$f(x) = \begin{cases}\dfrac{\sin[x]}{[x]}, & \text{if } x\in(-\infty,0)\cup[1,\infty)\\ 0, & \text{if } x\in[0,1)\end{cases}$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{\sin[-h]}{[-h]}$$

$$= \lim_{h\to 0}\frac{\sin(-1)}{(-1)} = \sin 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} 0 = 0$$

$$\therefore \lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$$

Therefore, $\lim_{x\to 0} f(x)$ does not exist.

5. d. $\lim_{x\to 0}\dfrac{\sqrt{\dfrac{1}{2}(1-\cos 2x)}}{x} = \lim_{x\to 0}\dfrac{\sqrt{\dfrac{1}{2}\cdot 2\sin^2 x}}{x} = \lim_{x\to 0}\dfrac{|\sin x|}{x}$

$$\therefore \text{L.H.L.} = \lim_{h\to 0}\frac{|\sin(0-h)|}{0-h} = \lim_{h\to 0}\frac{|-\sin h|}{-h} = \lim_{h\to 0}\frac{\sin h}{-h} = -1$$

$$\text{R.H.L.} = \lim_{h\to 0}\frac{|\sin(0+h)|}{0+h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

As L.H.L. ≠ R.H.L., the given limit does not exist.

6. d. L.H.L. $= \lim_{x\to 1^-}\dfrac{\sqrt{1-\cos[2(x-1)]}}{x-1}$

$$= \lim_{x\to 1^-}\frac{\sqrt{2\sin^2(x-1)}}{x-1}$$

$$= \sqrt{2}\lim_{x\to 1^-}\frac{|\sin(x-1)|}{x-1}$$

$$= \sqrt{2}\lim_{h\to 0}\frac{|\sin(-h)|}{-h} = \sqrt{2}\lim_{h\to 0}\frac{\sin h}{-h} = -\sqrt{2}$$

$$\text{R.H.L.} = \lim_{x\to 1^+}\sqrt{2}\frac{|\sin(x-1)|}{x-1}$$

$$= \lim_{h\to 0}\sqrt{2}\frac{|\sin h|}{h}$$

$$\lim_{h\to 0}\sqrt{2}\frac{\sin h}{h} = \sqrt{2}$$

L.H.L. ≠ R.H.L. Therefore, $\lim_{x\to 1} f(x)$ does not exist.

7. b. Putting $\theta = 0$, we get $b_0 = 0$

$$\therefore \quad \sin n\theta = \sum_{r=1}^{n} b_r \sin^r\theta$$

$$\Rightarrow \quad \frac{\sin n\theta}{\sin\theta} = \sum_{r=1}^{n} b_r(\sin\theta)^{r-1}$$

$$= b_1 + b_2\sin\theta + b_3\sin^2\theta + \ldots + b_n\sin^{n-1}\theta$$

Taking limit as $\theta\to 0$, we obtain

$$\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = b_1 \Rightarrow b_1 = n.$$

8. c. $\lim_{x\to 0}\dfrac{x\tan 2x - 2x\tan x}{4\sin^4 x} = \lim_{x\to 0}\dfrac{x}{4\sin^4 x}\left[\dfrac{2\tan x}{1-\tan^2 x} - 2\tan x\right]$

$$= \lim_{x\to 0} \frac{x\tan^3 x}{2\sin^4 x\left(1-\tan^2 x\right)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{x}{\sin x}\frac{1}{\cos^3 x}\frac{1}{1-\tan^2 x}$$

$$= \frac{1}{2}\times 1\times\frac{1}{1^3}\times\frac{1}{1-0} = \frac{1}{2}$$

9. c. $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x = e^{\lim_{x\to\infty}\left[\frac{x-3}{x+2}-1\right]x} = e^{\lim_{x\to\infty}\left[\frac{-5x}{x+2}\right]} = e^{-5}$

10. b. $\lim_{x\to 0}\frac{\sin\left(\pi\cos^2 x\right)}{x^2} = \lim_{x\to 0}\frac{\sin\left(\pi-\pi\cos^2 x\right)}{x^2}$

$[\sin(\pi-\theta)=\sin\theta]$

$$= \lim_{x\to 0}\frac{\sin\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\times\frac{\left(\pi\sin^2 x\right)}{x^2} = \pi$$

11. c. $L = \lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$= -\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)(\cos x-e^x)}{(1+\cos x)x^n}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2\left(\frac{1-\cos x}{x}+\frac{e^x-1}{x}\right)}{x^{n-3}}\frac{1}{1+\cos x}$$

L is finite nonzero. Then $n = 3$ (as for $n = 1, 2$, $L = 0$, and for $n = 4$, $L = \infty$).

12. d. Given $\lim_{x\to 0}\frac{\left[(a-n)nx-\tan x\right]\sin nx}{x^2}=0$, where a is non-zero number. Therefore,

$$n\lim_{x\to 0}\frac{\sin nx}{nx}\left[\left\{(a-n)n-\frac{\tan x}{x}\right\}\right]=0$$

or $n[(a-n)n-1]=0$

or $a = \frac{1}{n}+n$

13. c. $\lim_{x\to 0}\left[(\sin x)^{1/x}+(1/x)^{\sin x}\right]$

$$= \lim_{x\to 0}(\sin x)^{1/x}+\lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x}$$

$$= 0+e^{\lim_{x\to 0}\sin x\log\left(\frac{1}{x}\right)}$$

$$= e^{\lim_{x\to 0}\frac{-\log x}{\operatorname{cosec} x}} = e^{\lim_{x\to 0}\frac{-1/x}{-\operatorname{cosec} x\cot x}}$$ [Using L'Hospital's rule]

$$= e^{\lim_{x\to 0}\frac{\sin x}{x}\tan x} = e^0 = 1$$

14. c. $|x-1| = 1-x,\ x<1$

$\therefore p$ = left hand derivative of $|x-1|$ at $x = 1 = -1$

$\Rightarrow \lim_{x\to 1^+} g(x) = -1$

$\Rightarrow \lim_{h\to 0} g(1+h) = -1$

$\Rightarrow \lim_{h\to 0}\left(\frac{h^n}{\log\cos^m h}\right) = -1$

$\Rightarrow \lim_{h\to 0}\left(\frac{h^n}{m\log\cos h}\right) = -1$

$\Rightarrow \lim_{h\to 0}\frac{n\cdot h^{n-1}}{m\cdot(-\tan h)} = -1$ (Applying L'Hospital's Rule)

$\Rightarrow -\left(\frac{n}{m}\right)\lim_{h\to 0}\left(\frac{h^{n-1}}{\tan h}\right) = -1$

which holds if $n = m = 2$.

15. d. $\lim_{x\to 0}[1+x\ln(1+b^2)]^{1/x} = 2b\sin^2\theta$

$\Rightarrow e^{\lim_{x\to 0}[1+x\ln(1+b^2)-1]\frac{1}{x}} = 2b\sin^2\theta$

$\Rightarrow e^{\lim_{x\to 0}\ln(1+b^2)} = 2b\sin^2\theta$

$\Rightarrow e^{\ln(1+b^2)} = 2b\sin^2\theta$

$\Rightarrow 1+b^2 = 2b\sin^2\theta$

$\Rightarrow b+\frac{1}{b} = 2\sin^2\theta$

Since $b > 0$,

$$2\sin^2\theta = b+\frac{1}{b}\geq 2$$

$\Rightarrow \sin^2\theta\geq 1$

$\Rightarrow \sin^2\theta = 1$

$\Rightarrow \theta = \pm\pi/2$

16. b. $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right) = 4$

$\Rightarrow \lim_{x\to\infty}\left(x+\frac{1}{x+1}-ax-b\right) = 4$

$\Rightarrow \lim_{x\to\infty}((1-a)x-b) = 4$

$\Rightarrow 1-a = 0$ and $-b = 4$

$\Rightarrow a = 1$ and $b = -4$

17. b. Let $1+a = y$

$\Rightarrow (y^{1/3}-1)x^2+(y^{1/2}-1)x+y^{1/6}-1 = 0$

$\Rightarrow \left(\frac{y^{1/3}-1}{y-1}\right)x^2+\left(\frac{y^{1/2}-1}{y-1}\right)x+\frac{y^{1/6}-1}{y-1} = 0$

Now, taking $\lim_{y\to 1}$ on both sides, we get

$$\frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6} = 0$$

$\Rightarrow 2x^2+3x+1 = 0$

$$x = -1,\ -\frac{1}{2}$$

Multiple Correct Answers Type

1. a., c. $L = \lim_{x\to 0} \dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$

$$= \lim_{x\to 0} \frac{1}{x^2\left(a + \sqrt{a^2 - x^2}\right)} - \frac{1}{4x^2}$$

$$= \lim_{x\to 0} \frac{(4 - a) - \sqrt{a^2 - x^2}}{4x^2\left(a + \sqrt{a^2 - x^2}\right)}$$

Numerator $\to 0$ if $a = 2$ and then $L = \dfrac{1}{64}$.

Alternative Method:

$$L = \lim_{x\to 1} \frac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 1} \frac{a - (a^2 - x^2)^{\frac{1}{2}} - \dfrac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 1} \frac{a - a\left(1 - \dfrac{x^2}{a^2}\right)^{\frac{1}{2}} - \dfrac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 1} \frac{a - a\left(1 + \dfrac{1}{2}\left(-\dfrac{x^2}{a^2}\right) + \dfrac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(-\dfrac{x^2}{a^2}\right)^2\right) - \dfrac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 1} \frac{\left(\dfrac{1}{2a} - \dfrac{1}{4}\right)x^2 + \dfrac{x^4}{8a^3}}{x^4}$$

This limit exists if $\dfrac{1}{2a} - \dfrac{1}{4} = 0$ or $a = 2$.

Also, when $a = 2$ then

$$L = \lim_{x\to 1} \frac{\dfrac{x^4}{64}}{x^4} = \frac{1}{64}$$

Integer Answer Type

1. (2) $\lim_{x\to 1} \left\{\dfrac{-ax + \sin(x - 1) + a}{x + \sin(x - 1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \dfrac{1}{4}$

$$\Rightarrow \quad \lim_{x\to 1} \left(\frac{\dfrac{\sin(x - 1)}{(x - 1)} - a}{\dfrac{\sin(x - 1)}{(x - 1)} + 1}\right)^{1+\sqrt{x}} = \frac{1}{4}$$

$$\Rightarrow \quad \left(\frac{1 - a}{2}\right)^2 = \frac{1}{4}$$

$\Rightarrow \quad a = 0, a = 2$

2. (2) $m \geq 2$ and $n \geq 2$

$$\lim_{\alpha\to 0} \frac{e^{\cos(\alpha^n)} - e}{\alpha^m}$$

$$= \lim_{\alpha\to 0} \frac{e(e^{\cos(\alpha^n)-1} - 1)}{\cos(\alpha^n) - 1} \times \left(\frac{\cos(\alpha^n) - 1}{(\alpha^n)^2}\right)\frac{\alpha^{2n}}{\alpha^m}$$

$$= e \times \lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1}\right) \times \lim_{\alpha\to 0}\left(\frac{\cos(\alpha^n) - 1}{\alpha^{2n}}\right) \times \lim_{\alpha\to 0} \alpha^{2n-m}$$

$$= e \times 1 \times \lim_{\alpha\to 0} \frac{-2\sin^2 \dfrac{\alpha^n}{2}}{\alpha^{2n}} \times \lim_{\alpha\to 0} \alpha^{2n-m}$$

$$= e \times 1 \times \left(-\frac{1}{2}\right) \times \lim_{\alpha\to 0} \alpha^{2n-m}$$

Now, $\lim_{\alpha\to 0} \alpha^{2n-m}$ must be equal to 1.

i.e., $\quad 2n - m = 0$

or $\quad \dfrac{m}{n} = 2$

Fill in the Blanks Type

1. $\lim_{x\to 1} (1 - x) \tan \dfrac{\pi x}{2} = \lim_{x\to 1} \dfrac{(1 - x)}{\tan\left(\dfrac{\pi}{2} - \dfrac{\pi x}{2}\right)}$

$$= \frac{2}{\pi} \lim_{x\to 1} \frac{\dfrac{\pi}{2}(1 - x)}{\tan\left(\dfrac{\pi}{2}(1 - x)\right)}$$

$$= \frac{2}{\pi}$$

Alternative Method:

$$\lim_{x\to 1} (1 - x)\tan\frac{\pi x}{2}$$

$$= \lim_{x\to 1} \frac{(1 - x)}{\cot\dfrac{\pi x}{2}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{-1}{-\dfrac{\pi}{2}\operatorname{cosec}^2 \dfrac{\pi x}{2}} \qquad \text{(Applying L'Hospital's Rule)}$$

$$= \frac{2}{\pi}$$

2. $f(x) = \begin{cases} \sin x, & x \neq n\pi, n \in I \\ 2, & \text{otherwise} \end{cases}$

and $g(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 4, & x = 0 \\ 5, & x = 2 \end{cases}$

$\lim_{x\to 0^+} g\{f(x)\} = g(f(0^+)) = g((\sin 0^+)) = g(0^+) = (0)^2 + 1 = 1$

$\lim_{x\to 0^-} g\{f(x)\} = g(f(0^-)) = g((\sin 0^-)) = g(0^-) = (0)^2 + 1 = 1$

Hence, $\lim_{x\to 0} g\{f(x)\} = 1$.

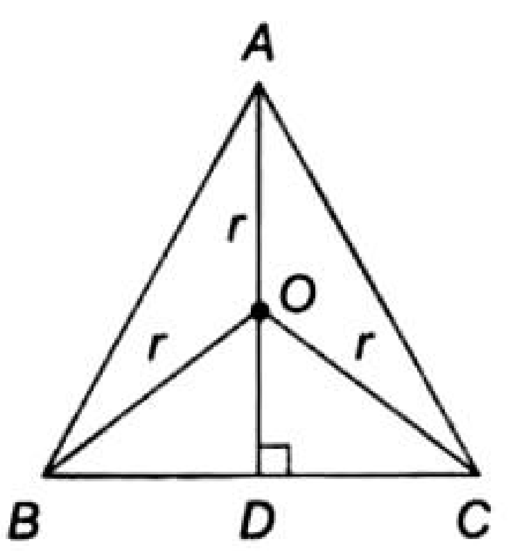
3. $\lim_{x\to-\infty}\left[\frac{x^4\sin\left(\frac{1}{x}\right)+x^2}{\left(1+|x|^3\right)}\right]=\lim_{x\to-\infty}\left[\frac{x\sin\left(\frac{1}{x}\right)+\frac{1}{x}}{\frac{1}{x^3}-1}\right]$

$$=\lim_{x\to-\infty}\left[\frac{\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}+\frac{1}{x}}{\frac{1}{x^3}-1}\right]=\frac{1+0}{0-1}=-1$$

4. In ΔABC, $AB = AC$, $AD \perp BC$ (D is the midpoint of BC).
Let r = radius of circumcircle
$\therefore OA = OB = OC = r$

Now, $BD=\sqrt{BO^2-OD^2}=\sqrt{r^2-(h-r)^2}$

$$=\sqrt{2rh-h^2}$$



$\therefore BC=2\sqrt{2rh-h^2}$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}\times BC\times AD=h\sqrt{2rh-h^2}$

Also, $\lim_{h\to0}\frac{A}{P^3}=\lim_{h\to0}\frac{h\sqrt{2rh-h^2}}{8\left(\sqrt{2rh-h^2}+\sqrt{2hr}\right)^3}$

$$=\lim_{h\to0}\frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}\left(\sqrt{2r-h}+\sqrt{2r}\right)^3}$$

$$=\lim_{h\to0}\frac{\sqrt{2r-h}}{8\left[\sqrt{2r-h}+\sqrt{2r}\right]^3}$$

$$=\frac{\sqrt{2r}}{8\left(\sqrt{2r}+\sqrt{2r}\right)^3}=\frac{\sqrt{2r}}{8\times8\times2r\times\sqrt{2r}}=\frac{1}{128r}$$

5. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}=\lim_{x\to\infty}\left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^{x+4}$

$$=\lim_{x\to\infty}\frac{\left(1+\frac{6}{x}\right)^x}{\left(1+\frac{1}{x}\right)^x}\left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^4\left[\text{Using }\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x=e^\lambda\right]$$

$$=\frac{e^6}{e}\left(\frac{1}{1}\right)^4=e^5$$

Alternative Method:

$\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$ $\left(1^\infty\text{ form as }\lim_{x\to\infty}\frac{x+6}{x+1}=\lim_{x\to\infty}\frac{1+\frac{6}{x}}{1+\frac{1}{x}}=1\right)$

$$=e^{\lim_{x\to\infty}\left(\frac{x+6}{x+1}-1\right)(x+4)}$$

$$=e^{\lim_{x\to\infty}\left(\frac{5(x+4)}{x+1}\right)}$$

$$=e^{\lim_{x\to\infty}5\left(\frac{1+\frac{4}{x}}{1+\frac{1}{x}}\right)}$$

$$=e^5$$

6. $\lim_{x\to0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}=\frac{\lim_{x\to0}\left(1+5x^2\right)^{1/x^2}}{\lim_{x\to0}\left(1+3x^2\right)^{1/x^2}}$

$$=\frac{\lim_{x\to0}\left\{\left(1+5x^2\right)^{\frac{1}{5x^2}}\right\}^5}{\lim_{x\to0}\left\{\left(1+3x^2\right)^{\frac{1}{3x^2}}\right\}^3}$$

$$=e^{5-3}=e^2$$

7. $\lim_{h\to0}\frac{\ln(1+2h)-2\ln(1+h)}{h^2}$

$$=\lim_{h\to0}\frac{\ln\left[\frac{(1+2h)}{1+2h+h^2}\right]}{h^2}$$

$$=\lim_{h\to0}\frac{\ln\left[1+\frac{-h^2}{1+2h+h^2}\right]}{\frac{-h^2}{1+2h+h^2}}\times\frac{-1}{1+2h+h^2}$$

$$=1\times\lim_{h\to0}\frac{-1}{1+2h+h^2}\qquad\left[\text{Using }\lim_{x\to0}\frac{\log(1+x)}{x}=1\right]$$

$$=-1$$

Alternative Method:

$$\lim_{h\to0}\frac{\ln(1+2h)-2\ln(1+h)}{h^2}$$

$$=\lim_{h\to0}\frac{\frac{2}{1+2h}-\frac{2}{1+h}}{2h}\quad\text{(Applying L'Hospital's Rule)}$$

$$=\lim_{h\to0}\frac{1+h-1-2h}{h(1+h)(1+2h)}$$

$$=\lim_{h\to0}\frac{-1}{(1+h)(1+2h)}$$

$$=-1$$

True/False Type

1. **False.** Consider $f(x) = \frac{|x-a|}{x-a}, g(x) = \frac{x-a}{|x-a|}$. Then, $\lim_{x\to a}(f(x) \times g(x))$ exists, but $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist. Therefore, statement is false.

Subjective Type

1. $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \lim_{x\to a} \frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})} = \frac{1}{3}\cdot\frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

Alternative Method:

$$\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to a} \frac{\frac{1}{\sqrt{a+2x}}-\frac{\sqrt{3}}{2\sqrt{x}}}{\frac{1}{2\sqrt{3a+x}}-\frac{1}{\sqrt{x}}} \quad \text{(Applying L'Hospital's Rule)}$$

$$= \frac{\frac{1}{\sqrt{3a}}-\frac{\sqrt{3}}{2\sqrt{a}}}{\frac{1}{2\sqrt{4a}}-\frac{1}{\sqrt{a}}}$$

$$= \frac{\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{2}}{\frac{1}{4}-1} = \frac{-\frac{1}{2\sqrt{3}}}{-\frac{3}{4}} = \frac{2}{3\sqrt{3}}$$

2. $f(x) = \int \frac{2\sin x - \sin 2x}{x^3}dx, x \neq 0$

$$\therefore f'(x) = \frac{2\sin x - \sin 2x}{x^3}, x \neq 0$$

$$\therefore \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3}$$

$$= \lim_{x\to 0} \frac{2\sin x(1-\cos x)(1+\cos x)}{x^3(1+\cos x)}$$

$$= \lim_{x\to 0} 2 \times \frac{\sin^3 x}{x^3} \times \frac{1}{1+\cos x} = 2\times(1)^3 \times \frac{1}{2} = 1$$

Alternative Method:

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x^3} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{3x^2} \quad \text{(Applying L'Hospital's Rule)}$$

$$= \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{6x} \quad \text{(Applying L'Hospital's Rule)}$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{6} \quad \text{(Applying L'Hospital's Rule)}$$

$$= 1$$

3. $\lim_{h\to 0} \frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}$

$$= \lim_{h\to 0} \frac{a^2[\sin(a+h)-\sin a] + 2ah\sin(a+h) + h^2\sin(a+h)}{h}$$

$$= \lim_{h\to 0} \frac{a^2\left[2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}\right]}{2\times\frac{h}{2}} + \lim_{h\to 0} 2a\sin(a+h) + \lim_{h\to 0} h\sin(a+h)$$

$$= a^2\cos a + 2a\sin a$$

Alternative Method:

$$\lim_{h\to 0} \frac{(a+h)^2\sin(a+h) - a^2\sin a}{h} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{h\to 0} \frac{2(a+h)\sin(a+h) + (a+h)^2\cos(a+h)}{1} \quad \text{(Applying L'Hospital's Rule)}$$

$$= 2a\sin a + a^2\cos a$$

4. $\lim_{x\to 0} \frac{2^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0} \frac{2^x-1}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$$= \lim_{x\to 0} \frac{(2^x-1)(\sqrt{1+x}+1)}{1+x-1}$$

$$= \lim_{x\to 0} \frac{2^x-1}{x} \lim_{x\to 0}(\sqrt{1+x}+1)$$

$$= \ln 2\,(1+1) = 2\ln 2$$

5. $\lim_{x\to 0} \left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x} = \lim_{x\to 0} \left\{\frac{1+\tan x}{1-\tan x}\right\}^{1/x}$

$$= \frac{\lim_{x\to 0}\left[(1+\tan x)^{1/\tan x}\right]^{\frac{\tan x}{x}}}{\lim_{x\to 0}\left[(1-\tan x)^{-1/\tan x}\right]^{-\frac{\tan x}{x}}}$$

$$= \frac{e}{e^{-1}} = e^2$$

Alternative Method:

$$\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x} \qquad (1^{\infty}\text{ form})$$

$$= e^{\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)-1\right\}\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left\{\frac{2\tan x}{1+\tan x}\right\}\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\frac{\tan x}{x}\cdot\frac{2}{1+\tan x}}$$

$$= e^{2}$$